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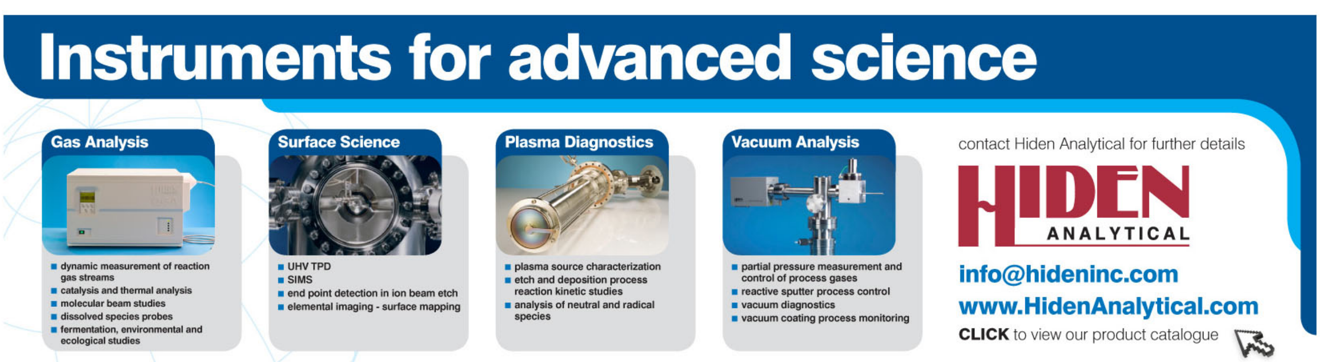

The two solutions according to Eqs. (5) and (6) are compared in Fig. 4 and seen to be in substantial agreement. If the number of nodal points is increased in the vicinity of the expected maximum temperature gradient, the two solutions become virtually identical.

The temperature profiles in Fig. 4 for variable heat generation were obtained numerically, and serve to point out the need for a method with sufficient flexibility to encompass the marked influence of this particular nonuniformity.

REMARKS

A numerical method for temperature fields in porous structures generating nonuniform heat has been presented and exemplified by several practical one-dimensional applications. Similar numerical solutions can also be effected for more general one- and two-dimensional cases encompassing the effects of temperature-dependent properties as well as the effect of heat generation dependent on local temperature.

While it is apparent that the actual physical situation encountered in porous heat transfer is substantially removed from that suggested by the general residual equation (1), the method does appear to give sufficiently accurate results to warrant its application in exploratory design of either generating or nongeneration porous structures.

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An Operational Analysis of Traffic Dynamics

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The dynamics of a line of traffic composed of n vehicles is studied mathematically. It is postulated that the movements of the several vehicles are controlled by an idealized "law of separation." The law considered in the analysis specifies that each vehicle must maintain a certain prescribed "following distance" from the preceding vehicle. This distance is the sum of a distance proportional to the velocity of the following vehicle and a certain given minimum distance of separation when the vehicles are at rest. By the application of this postulated law to the motion of the column of vehicles, the differential equations governing the dynamic state of the system are obtained.

The solution of the dynamical equations for several assumed types of motion of the leading vehicle is effected by the operational or Laplace transform method and the velocities and accelerations of the various vehicles are thus obtained. Consideration is given to the use of an electrical analog computer for studying the dynamical equations of the system.

I. INTRODUCTION

DURING the last few years there has been expressed considerable interest in the applications of the methods of "operations research" to the study of problems in many diversified fields of investigation.[1,2] The study of automobile traffic flow provides an excellent subject for the application of these techniques. Several types of investigations are being carried out in the field of traffic engineering. One is a statistical study of the behavior of traffic at intersections. Another field of investigation is the study of the dynamics of a line of traffic.[3,4,4a]

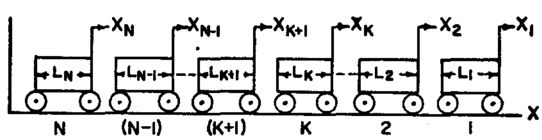

FIG. 1.

It has been found that when the light at an intersection turns green, the whole line of vehicles controlled by it does not begin to move as a unit but a wave of "starting" travels down the line of vehicles. Observations have disclosed the remarkable fact that the velocity of this wave has the approximate constant value of thirty miles per hour along the row of vehicles. The implications of these investigations[5–12] are that certain fundamental parameters involving driver characteristics may be discovered and incorporated into the design of motor vehicles and roads to improve traffic flow and prevent congestion.

The purpose of the present discussion is to develop a mathematical analysis of the dynamics of a line of traffic which results on assuming that the drivers of the various vehicles of the line at all times obey a postulated traffic regulation. This regulation is suggested by the following statement in the California Vehicle Code

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[1] P. M. Morse and G. E. Kimball, *Methods of Operations Research* (Technology Press and John Wiley and Sons, Inc., New York, 1950).

[2] P. M. Morse, J. Appl. Phys. **23**, 165 (1952).

[3] E. M. J. Herrey and H. Herrey, Am. J. Phys. **13**, 1 (1945).

[4] L. A. Pipes, "A proposed dynamic analogy of traffic," Institute of Transportation and Traffic Engineering, University of California, Los Angeles (July, 1950).

[4a] A. Reuschel, Öst. Ing. Arch. 4, 3, 193 (1950).

[5] R. Mayne, Elec. Eng. 207 (March, 1951).

[6] F. V. Taylor, "Certain characteristics of the human servo" Elec. Eng. **68** (March, 1949).

[7] J. A. V. Bates, J. Inst. Elec. Engrs. **94**, 298 (1947).

[8] K. J. W. Craik, Brit. J. Psych. **38**, 56, 142 (1947).

[9] D. G. Ellson, Psych. Rev. **56**, 9 (1949).

[10] W. E. Hicks, Quart. J. Exptl. Psych. **1**, 36 (1948).

[11] A. Tustin, J. Inst. Elec. Engrs. **94**, 190 (1947).

[12] Warren, Fitts, and Clark, "An electronic apparatus for the study of the human operator in a one-dimensional, closed-loop, continuous pursuit task," American Institute of Electrical Engineers Technical Paper 52–8 (November, 1951).

Summary. "A good rule for following another vehicle at a safe distance is to allow yourself the length of a car (about fifteen feet) for every ten miles per hour you are traveling." This regulation is based on the fact that thousands of tests (and accidents) have proved that as a consequence of human nervous characteristics and the many distractions encountered while driving, it can easily take one full second for the driver's eyes to see that the fellow ahead has clamped on his brakes and for the operator's brain to direct his right foot to get off the accelerator and put on the brakes. It is apparent that during this short period the vehicle will continue to travel without changing its speed and therefore a minimum of one second's reaction distance should be maintained between the vehicles.

The dynamics of a line of traffic developed in this discussion is based on the assumption that the driver of each vehicle of the line obeys a generalization of the above rule at all times. By the application of this postulated regulation to each vehicle of the line, the dynamical equations of the system are obtained. These equations are then solved by the Laplace transform or "operational" method for various specified motions of the leading vehicle. The possibilities of solving the dynamical equations of the system by the use of an electrical circuit are also discussed.

II. NOTATION; THE DYNAMICAL EQUATIONS

Consider the line of n vehicles moving to the right as shown schematically in Fig. 1. The following notation will be used in the analysis:

n = the number of vehicles in the line of traffic.
k = an index number.
t = the time in seconds.
x_k = the coordinate of the front of the kth vehicle (feet) in an arbitrary Cartesian reference frame.
L_k = the length of the kth vehicle (feet).
$v_k=\dot{x}_k$ = the velocity of vehicle # k. (ft/sec).
$a_k=\dot{v}_k=\ddot{x}_k$ = the acceleration of vehicle # k. (ft/sec^2).
b = the prescribed legal distance between the vehicles at standstill (ft).
T = a time constant (seconds) prescribed by the postulated "traffic law." ($T=15.00/14.67=1.02$ sec, California Vehicle Code.)
Tv_k = the legal "Speedometer distance of separation of the kth and $(k-1)$th vehicles."
$(b+Tv_{k+1})$ = the postulated legal distance between the $(k+1)$th and the kth vehicles.
S_k = the distance moved by car # k in time t (feet).

$$=\int_0^t v_k(t)dt.$$

$$\dot{v}_{k+1}=\frac{1}{T}(v_k-v_{k+1}),\quad k=1,2,3,\cdots(n-1).$$

(The set of dynamical equations of the line of traffic.)

$$Q(k)=\int_0^\infty e^{-u}u^{(k-1)}du=(k-1)!,\quad k=1,2,3,4\cdots$$

= the complete gamma-function.

$$Q_t(k)=\int_0^t e^{-u}u^{(k-1)}du=\text{the incomplete gamma-function.}$$

$$G_k(t)=\frac{Q_t(k)}{Q(k)}=\text{the ratio of the incomplete to the complete gamma-function.}$$

In the analysis, the quantity Tv_{k+1} will be termed "the speedometer distance" between the vehicles k and $(k+1)$. The parameter T has dimensions of time, and its magnitude may be determined from the postulated "law of separation" of the vehicles. If the separation law of the California Vehicle Code is taken as an example, then the "speedometer distance" Tv_{k+1} has the magnitude of 15 ft when the vehicle speed is ten miles per hour or 14.67 ft/sec. Hence, to determine T, we have the equation

$$Tv_{k+1}=T(14.67)=15.00,\tag{2.1}$$

or

$$T=15.00/14.67=1.023\text{ seconds.}\tag{2.2}$$

In the analysis it is convenient to take the value of one second for T.

The Dynamical Equations of the Line of Traffic

The dynamical equations that govern the line of traffic may be obtained by requiring that each vehicle shall be separated by the postulated "legal distance." By reference to Fig. 1, it can be seen that when the "traffic law" is obeyed, the coordinates, x_{k+1} and $x_k{}^a$ of two successive vehicles of the line must satisfy the following set of equations:

$$x_k=x_{k+1}+(b+Tv_{k+1})+L_k\qquad k=1,2,3,4,\cdots(n-1).\tag{2.3}$$

If these equations are differentiated with respect to time, the result is,

$$\dot{x}_k=\dot{x}_{k+1}+T\dot{v}_{k+1},\quad k=1,2,3,4,\cdots(n-1).\tag{2.4}$$

These equations may be written in terms of the velocities of the vehicles in the form

$$T\dot{v}_{k+1}+v_{k+1}=v_k,\quad k=1,2,3,4,\cdots(n-1).\tag{2.5}$$

Equations (2.5) are the dynamical equations of the system of vehicles. In order to solve them, let the p multiplied Laplace transform of the velocity, $v_k(t)$ be introduced by the equation,[13]

$$Lv_k(t) = V_k(p). \tag{2.6}$$

By the transformation formula for the first derivative of a function, we have

$$L\dot{v}_k(t) = pV_k(p) - pv_k(0), \tag{2.7}$$

where the $v_k(0)$ quantities are the *initial* velocities of the vehicles at $t=0$. By the introduction of the transforms (2.6) and (2.7), the differential equations (2.5) are transformed into the following set of algebraic equations:

$$(Tp+1)V_{k+1} = V_k + Tpv_{k+1}(0) \quad \text{for} \quad k=1, 2, 3, 4, \cdots (n-1). \tag{2.8}$$

This set of equations is fundamental in the study of the motion of the line of vehicles under the postulated "traffic law" and will be used to study special cases of practical significance.

III. THE MOVEMENT OF A LINE OF VEHICLES INITIALLY AT REST

The simplest type of motion to analyze is the one in which it is desired to determine the subsequent motion of the various vehicles of the line of traffic when it is initially at rest at $t=0$, and the motion of the leading vehicle is specified for $t>0$. This case simulates the practical situation in which a line of n vehicles is standing still and waiting for the change of a traffic signal. It is assumed that the traffic signal changes from "stop" to "go" at $t=0$, and that the leading vehicle begins to move forward after $t=0$ with a specified known velocity given by

$$v_1(t) = F(t) \quad \text{for} \quad t>0. \tag{3.1}$$

Let the Laplace transform of the velocity of the leading vehicle be given by

$$Lv_1(t) = LF(t) = V_1(p). \tag{3.2}$$

Since $v_1(t)$ is assumed known for $t>0$, $V_1(p)$ is also known. Because the vehicles are assumed initially at rest, we have zero initial velocities and hence

$$v_{k+1}(0) = 0, \quad \text{for} \quad k=1, 2, 3, 4, \cdots (n-1). \tag{3.3}$$

Therefore, the fundamental equations (2.8) in this case may be written in the following form:

$$V_{k+1} = \frac{V_k}{(Tp+1)}, \quad k=1, 2, 3, 4, \cdots (n-1). \tag{3.4}$$

By substituting the value of V_k given by the kth equation into the $(k+1)$th equation, one obtains

$$V_{k+1} = \frac{V_1}{(Tp+1)^k}, \quad k=1, 2, \cdots (n-1). \tag{3.5}$$

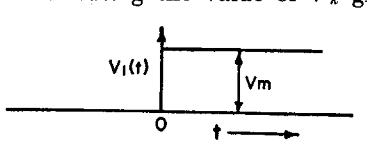


FIG. 2.

[13] L. A. Pipes, *Applied Mathematics for Engineers and Physicists.* (McGraw-Hill Book Company, Inc., New York, 1946), pp. 119–140.

This set of equations relates the transform of the velocity of each vehicle to the known transform of the velocity of the first vehicle. A formal solution that gives the velocity of every vehicle of the line in terms of the given velocity of the leading vehicle $v_1(t)$ may be obtained by the use of the Faltung or superposition theorem of the Laplace transform theory.[13] The velocity of each vehicle may be expressed in the symbolic form

$$v_{k+1}(t) = L^{-1}V_{k+1}(p) = L^{-1}\left[\frac{V_{1(p)}}{(Tp+1)^k}\right] \quad \text{for } k=1, 2, 3, 4, \cdots (n-1). \tag{3.6}$$

The Faltung or superposition theorem states that, under suitable conditions, if

$$g_1(p) = Lh_1(t) \tag{3.7}$$

and

$$g_2(p) = Lh_2(t), \tag{3.8}$$

then

$$L^{-1}\left[\frac{g_1(p)g_2(p)}{p}\right] = \int_0^t h_1(u)h_2(t-u)du. \tag{3.9}$$

To compute the inverse transform of (3.7), let

$$g_1(p) = \frac{p}{(Tp+1)^k} \tag{3.10}$$

and

$$g_2(p) = V_1(p) = LF(t). \tag{3.11}$$

Now from a table of Laplace transforms,[13] we have

$$L^{-1}g_1(p) = L^{-1}\left[\frac{p}{(Tp+1)^k}\right] = \frac{t^{k-1}}{T^k} \cdot \frac{e^{-t/T}}{(k-1)!} = h_1(t) \tag{3.12}$$

and, by hypothesis, we have

$$L^{-1}g_2(p)L^{-1}V_1(p) = h_2(t) = F(t). \tag{3.13}$$

Hence, on applying the Faltung theorem to (3.7), the following result is obtained:

$$v_{k+1}(t) = \left[\frac{T^{-k}}{(k-1)!}\right] \times \int_0^t u^{(k-1)} \exp(-u/T)F(t-u)du \quad \text{for} \quad k=1, 2, 3, 4, \cdots (n-1). \tag{3.14}$$

This formal solution gives the velocities of the various vehicles of the line of traffic when the velocity $F(t)$ of the leading vehicle is given. However, to obtain the solution of special cases of practical significance it is simpler to proceed from (3.7) directly rather than by the use of the general equation (3.15). Three special cases will now be considered.

IV. IMPULSE ACCELERATION OF THE LEADING VEHICLE

The simplest case from a mathematical standpoint is the one in which it is assumed that the leading vehicle is standing still and suddenly acquires a constant velocity v_m at $t=0$. This case is, of course, not a physically realizable one since it requires the leading vehicle to undergo an impulsive acceleration of the delta-function type. The velocity of the leading vehicle is now assumed to be a step function as shown in Fig. 2. Because of its mathematical simplicity and because it is fundamental in the study of other more realistic cases an analysis of this case will be given.

The transform of the velocity of the leading vehicle is now,

$$Lv_1(t)=v_m, \tag{4.1}$$

where v_m is the magnitude of the step function velocity of the leading vehicle for $t>0$.

The transforms of the velocities of the subsequent vehicles of the line are obtained by substituting (4.1) into (3.6). They have the form

$$V_{k+1}(p)=\frac{v_m}{(Tp+1)^k}, \quad \text{for} \quad k=1,2,3,4,\cdots(n-1). \tag{4.2}$$

For simplicity, the time parameter T will be taken to be one second. This does not entail any loss in generality because as may be seen from the entry No. 2 of the Table of Transforms (see Appendix), the solution for any other value of T is the same as that for $T=1$, if the unit of time is chosen to be equal to T. If this is done, the velocities of the various vehicles are given by the equations,

$$v_{k+1}(t)=L^{-1}\frac{v_m}{(p+1)^k}, \quad \text{for} \quad k=1,2,3,4,\cdots(n-1). \tag{4.3}$$

The function $(p+1)^{-k}$, for $k>0$, has the well-known transform[13]

$$L^{-1}(p+1)^{-k}=\frac{1}{Q(k)}\int_0^t e^{-u}u^{(k-1)}du, \quad k>0 \tag{4.4}$$

where

$$Q(k)=\int_0^\infty e^{-u}u^{(k-1)}du, \quad k>0 \tag{4.5}$$

is the *complete gamma-function.*[14] For positive integer values of k, we have

$$Q(k+1)=k!, \quad \text{the factorial.} \tag{4.6}$$

The integral,

$$Q_t(k)=\int_0^t e^{-u}u^{(k-1)}du, \quad k>0 \tag{4.7}$$

has been the subject of considerable investigation,[15–17]

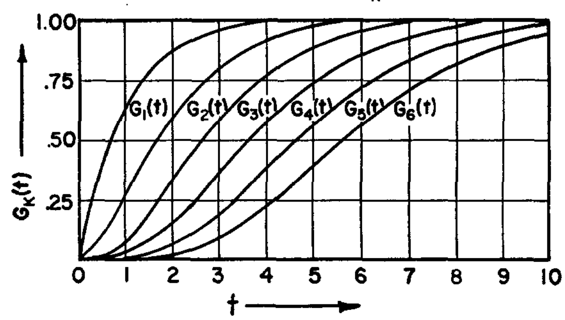


FIG. 3.

and has been termed the incomplete gamma-function. For $t=\infty$, it becomes identical with the complete gamma-function $Q(k)$. It is apparent from (4.4) and (4.7) that,

$$L^{-1}(p+1)^{-k}=\frac{Q_t(k)}{Q(k)}=G_k(t), \quad k>0. \tag{4.8}$$

The function $G_k(t)$ has the properties that,

$$G_k(0)=0, \quad G_k(\infty)=1, \quad \text{for} \quad k>0. \tag{4.9}$$

If k is a positive integer, the integral in (4.4) may be evaluated by parts to give the result

$$G_k(t)=1-e^{-t}\left[1+\frac{t}{1!}+\frac{t^2}{2!}+\cdots+\frac{t^{(k-1)}}{(k-1)!}\right]. \tag{4.10}$$

A short table of the functions $G_k(t)$ for $k=1$ to 6 and t from 0 to 10 is given in the Appendix. These functions are illustrated graphically by Fig. 3. It can be shown that the functions $G_k(t)$ satisfy the following recursion formula:

$$G_k(t)-G_{k+1}(t)=e^{-t}t^k/k!=G_{k+1}'(t), \quad k>0. \tag{4.11}$$

As a consequence of (4.3), the velocities of the various following vehicles produced by a step-function velocity of the leading vehicle are given by

$$v_{k+1}=v_mG_k(t), \quad k=1,2,3,4,\cdots(n-1). \tag{4.12}$$

The variations of the velocities of the various vehicles are therefore proportional to the curves of Fig. 3.

The accelerations of the vehicles are given by,

$$a_{k+1}=L^{-1}v_m\frac{p}{(p+1)^k} \quad k=1,2,3,4,\cdots(n-1)$$
$$=\frac{t^{(k-1)}e^{-t}}{(k-1)!}. \tag{4.13}$$

The distances traveled by the vehicles at a time t after the first vehicle has begun to move are

$$S_{k+1}=\int_0^t v_{k+1}(t)dt=v_m\int_0^t G_k(t)dt. \tag{4.14}$$

[14] See reference 13, chapter 12.

[15] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1927).

[16] K. Pearson, *Tables of the Incomplete Gamma Functions* (London, 1922).

[17] K. Pearson, *Tables for Statisticians and Biometricians* **1**, (London, 1930), third edition.

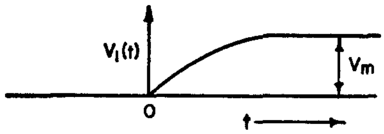


FIG. 4.

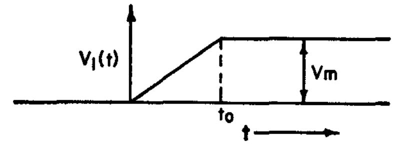


FIG. 5.

V. EXPONENTIAL ACCELERATION OF THE LEADING VEHICLE

A more realistic type of acceleration is one in which the final cruising velocity v_m of the leading vehicle is attained by a period of gradual acceleration of the type

$$a_1(t)=\dot{v}_1=cv_me^{-ct}, \tag{5.1}$$

where c is a constant having dimensions of $\sec^{-1}$. Since it is supposed that at $t=0$ the vehicle is standing still, this leads to the following variation of velocity:

$$v_1=v_m(1-e^{-ct}). \tag{5.2}$$

This variation of the velocity of the leading vehicle is illustrated by Fig. 4. The Laplace transform of the velocity of the leading vehicle is

$$V_1(p)=Lv_1(t)=v_mc/(p+c). \tag{5.3}$$

The Laplace transforms of the velocities of the various following vehicles may be obtained by means of (3.6). They have the form

$$V_{k+1}=\frac{v_mc}{(p+c)(Tp+1)^k},\quad k=1,2,3,4,\cdots(n-1). \tag{5.4}$$

For simplicity, take the constants c and T equal to unity. Then (5.4) reduces to

$$V_{k+1}=\frac{v_m}{(p+1)^{k+1}},\quad k=1,2,3,4,\cdots(n-1). \tag{5.5}$$

The inverse transform of this equation gives the following expression for the velocities of the following vehicles:

$$v_{k+1}(t)=v_mG_{k+1}(t). \tag{5.6}$$

It is thus seen that the velocities of the several vehicles are again proportional to the ordinates of the curves of Fig. 3.

VI. CONSTANT ACCELERATION OF THE LEADING VEHICLE

Another special case that yields to a simple mathematical analysis is the one in which the leading vehicle begins to move with a constant acceleration at $t=0$, until it reaches its cruising velocity T_0 seconds later. Figure 5 illustrates the assumed type of velocity variation of the first vehicle.

The Laplace transform of this function is

$$Lv_1(t)=\frac{v_m(1-e^{-pT_0})}{pT_0}=V_1(p). \tag{6.1}$$

The Laplace transforms of the velocities of the following vehicles of the line are obtained by substituting (6.1) into (3.6) and obtaining

$$V_{k+1}=\frac{v_m(1-e^{-pT_0})}{pT_0(Tp+1)^k},\quad k=1,2,3,4,\cdots(n-1). \tag{6.2}$$

As a consequence of entries No. 2 and No. 8 of the Table of Transforms (see Appendix) we have

$$L^{-1}\frac{1}{(Tp+1)^k}=G_k(t/T),\quad k>0. \tag{6.3}$$

By the use of No. 3 of the Table of Transforms and the integration theorem of the Laplace transform theory,[13] we have

$$v_{k+1}(t)=L^{-1}V_{k+1}(p)=\frac{v_m}{T_0}\int_0^t G_k(t/T)dt,\quad 0<t<T_0 \tag{6.4}$$

and

$$v_{k+1}(t)=\frac{v_m}{T_0}\left[\int_0^t G_k(t/T)dt-\int_0^{(t-T_0)} G_k(u/T)du\right],$$
$$t>T_0,\quad \text{for } k=1,2,3,4,\cdots(n-1). \tag{6.5}$$

Equations (6.4) and (6.5) are the velocities of the following vehicles before and after the leading vehicle has reached cruising speed.

The transforms of the accelerations of the various vehicles are

$$La_{k+1}(t)=\frac{v_m}{T_0}\frac{(1-e^{-pT_0})}{(Tp+1)^k},\quad k=1,2,3,4,\cdots(n-1). \tag{6.6}$$

The inverse transform of (6.6) may be obtained by No. 8, No. 2, and No. 3 of the Table of Transforms given in the Appendix. Hence, the accelerations of the various vehicles are

$$a_{k+1}(t)=\frac{v_m}{T_0}G_k(t/T),\quad 0<t<T_0,$$
$$k=1,2,3,4,\cdots(n-1). \tag{6.7}$$

and

$$a_{k+1}(t)=\frac{v_m}{T_0}\left[G_k(t/T)-G_k\left(\frac{t-t_0}{T}\right)\right],\quad t>T_0,$$
$$k=1,2,3,\cdots(n-1). \tag{6.8}$$

Equations (6.7) and (6.8) give the accelerations of the following vehicles before and after the leading vehicle has assumed cruising speed.

VII. GENERAL EQUATIONS GOVERNING THE DECELERATION OF THE LINE OF TRAFFIC

The general equations that govern the deceleration of the line of traffic under the operation of the postulated "traffic law" will now be considered. It will be assumed that at $t<0$ all the vehicles of the line are moving with the same cruising speed v_0 and that at $t>0$ the leading vehicle begins to decelerate because it approaches a "stop light" or some other impediment to its progress. The functional dependence of the deceleration of the first vehicle with the time is assumed to be known, and it is desired to obtain the velocities and decelerations of the other vehicles after the leading one has begun to decelerate. In order to obtain the transforms of the velocities of the various vehicles under the conditions of deceleration, we return to Eqs. (2.8) and place

$$v_{k+1}(0)=v_0 \quad \text{for} \quad k=1,2,3,4,\cdots(n-1). \tag{7.1}$$

Equation (2.8) may then be written in the following form:

$$v_{k+1}=\frac{1}{(Tp+1)}[V_k+Tpv_0], \quad k=1,2,3,4,\cdots(n-1). \tag{7.2}$$

This equation may be used to relate V_2, V_3, V_4, etc., to the transform of the velocity of the leading vehicle V_1 and the following equations are obtained:

$$\begin{aligned}
V_2&=\frac{1}{(Tp+1)}[V_1+Tpv_0],\\
V_3&=\frac{1}{(Tp+1)^2}[V_1+Tpv_0]+\frac{Tpv_0}{(Tp+1)},\\
V_4&=\frac{1}{(Tp+1)^3}[V_1+Tpv_0]+\frac{Tpv_0}{(Tp+1)^2}+\frac{Tpv_0}{(Tp+1)},\\
V_5&=\frac{1}{(Tp+1)^4}[V_1+Tpv_0]\\
&\quad+\frac{Tpv_0}{(Tp+1)^3}+\frac{Tpv_0}{(Tp+1)^2}+\frac{Tpv_0}{(Tp+1)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
V_{k+1}&=\frac{1}{(Tp+1)^k}[V_1+Tpv_0]+\Big[\frac{1}{(Tp+1)^{k-1}}\\
&\quad+\frac{1}{(Tp+1)^{k-2}}+\cdots+\frac{1}{(Tp+1)}\Big]Tpv_0.
\end{aligned} \tag{7.3}$$

Equations (7.3) are the general equations for the transforms of the velocities of the vehicles of the line of traffic in terms of the transform of the velocity of the leading vehicle during the period of deceleration. Two different types of deceleration of the leading vehicle will now be considered.

VIII. THE MOVEMENT OF THE LINE OF TRAFFIC AFTER A SUDDEN STOP OF THE LEADING VEHICLE

A special case of practical importance is the one in which the leading vehicle is initially moving with a velocity of v_0 along with the entire line of traffic at $t<0$ and at $t=0$, it comes suddenly to an abrupt stop. This type of motion of the leading vehicle simulates the effect of a collision on the movement of the line of traffic. The velocity of the leading vehicle is represented graphically as a function of time by the step-function of Fig. 6. The transform of the velocity of the leading vehicle is

$$Lv_1(t)=V_1(p)=0. \tag{8.1}$$

Hence, the system of equations (7.4) now reduces to

$$\begin{aligned}
V_2&=\frac{Tpv_0}{(Tp+1)},\\
V_3&=\left[\frac{1}{(Tp+1)^2}+\frac{1}{(Tp+1)}\right]Tpv_0,\\
V_4&=\left[\frac{1}{(Tp+1)^3}+\frac{1}{(Tp+1)^2}+\frac{1}{(Tp+1)}\right]Tpv_0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
V_{k+1}&=\left[\frac{1}{(Tp+1)^k}+\frac{1}{(Tp+1)^{k-1}}+\cdots+\frac{1}{(Tp+1)}\right]Tpv_0.
\end{aligned} \tag{8.2}$$

For simplicity let T equal one second since no loss of generality is lost in so doing. Now by No. 11 of the Table of Transforms (see Appendix), we have

$$L^{-1}\frac{p}{(p+1)^k}=\frac{e^{-t}t^{(k-1)}}{(k-1)!}=G_k'(t)=\Phi_k(t). \tag{8.3}$$

This result may be used to express the inverse transforms of Eqs. (8.2) with $T=1$, in the following form:

$$\begin{aligned}
v_2&=v_0\Phi_1(t)\\
v_3&=v_0[\Phi_2(t)+\Phi_1(t)]\\
v_4&=v_0[\Phi_3(t)+\Phi_2(t)+\Phi_1(t)]\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
v_{k+1}&=v_0[\Phi_k(t)+\Phi_{k-1}(t)+\Phi_{k-2}(t)+\cdots+\Phi_1(t)].
\end{aligned} \tag{8.4}$$

This gives the distribution of velocities of the various vehicles for $t>0$, after the first vehicle has come to an abrupt stop. The variation of the velocities of the first five vehicles is shown graphically in Fig. 6.

The decelerations of the various vehicles may be computed most simply by writing Eq. (2.5) in the form

$$a_{k+1}(t)=\frac{1}{T}[v_k-v_{k+1}], \quad \text{for } k=1,2,3,4,\cdots(n-1). \tag{8.5}$$

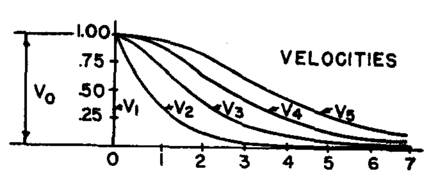


FIG. 6.

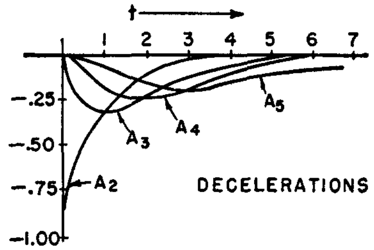


FIG. 7.

If T is taken to be one second, the deceleration of the second vehicle is given by

$$a_2(t)=(v_1-v_2)=-v_0\Phi_1(t)=-v_0e^{-t}. \qquad (8.6)$$

Under this assumption, the maximum deceleration of the second vehicle is seen to have the magnitude of v_0. For example, if the line of vehicles were moving at fifty miles per hour, the maximum deceleration of the second vehicle would have to be 73.4 ft/sec^2 in order to satisfy the postulated traffic regulation. With T equal to one second, the deceleration of the third vehicle is given by (8.5) in the form

$$a_3(t)=(v_2-v_3)=-v_0\Phi_2(t)=-v_0te^{-t}. \qquad (8.7)$$

As a consequence of (8.5) and (8.4) the deceleration of the $(k+1)$th vehicle of the line is given by

$$a_{k+1}(t)=-v_0\Phi_k(t), \quad k=1,2,3,4,\cdots(n-1). \quad (8.8)$$

The variations of the decelerations of the various vehicles as functions of time are shown in Fig. 7.

For $T=1$ second, the distances traveled by the vehicles during the period of deceleration are given by

$$S_{k+1}(t)=\int_0^t v_{k+1}(t)dt =v_0[G_k(t)+G_{k-1}(t)+\cdots+G_1(t)]. \quad (8.9)$$

This result follows as a consequence of Eqs. (8.3) and (8.4).

IX. EXPONENTIAL DECELERATION OF THE LEADING VEHICLE

An interesting case of practical importance is the one in which the leading vehicle undergoes an exponential acceleration of the form

$$a_1(t)=\dot{v}_1=-cv_0e^{-ct}, \qquad (9.1)$$

where c is a constant whose dimensions are 1/sec. The velocity of the leading vehicle for $t>0$ is now given by the equation

$$v_1(t)=v_0e^{-ct}. \qquad (9.2)$$

The transform of the velocity of the leading vehicle is

$$Lv_1(t)=\frac{pv_0}{(p+c)}=V_1(p). \qquad (9.3)$$

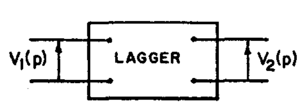


FIG. 8.

CASCADE CONNECTION OF LAGGERS

FIG. 9.

If this is substituted into (7.4), the following transform for the velocity of the $(k+1)$th vehicle is obtained:

$$V_{k+1}=\frac{1}{(Tp+1)^k}\left[\frac{pv_0}{(p+c)}+Tpv_0\right]+\left[\frac{1}{(Tp+1)^{k-1}}+\frac{1}{(Tp+1)^{k-2}}+\cdots+\frac{1}{(Tp+1)}\right]Tpv_0. \quad (9.4)$$

For simplicity, let the numerical values of the constants c and T be taken to be equal to unity. Then (9.4) reduces to

$$V_{k+1}=\left[\frac{1}{(p+1)^{k+1}}+\frac{1}{(p+1)^k}+\frac{1}{(p+1)}\right]pv_0 \quad \text{for} \quad k=1,2,3,4,\cdots(n-1). \quad (9.5)$$

The inverse transform of (9.5) may be computed by means of (8.3) and it has the form

$$v_{k+1}=v_0[\Phi_{k+1}(t)+\Phi_k(t)+\cdots+\Phi_1(t)]. \qquad (9.6)$$

This gives the velocity of the $(k+1)$th vehicle during the period of deceleration. The deceleration of the various vehicles in this case has the form

$$a_{k+1}(t)=-v_0\Phi_{k+1}(t), \quad k=1,2,3,4,\cdots(n-1). \quad (9.7)$$

The distance traveled by the vehicles during the period of deceleration is

$$S_{k+1}(t)=v_0[G_{k+1}(t)+G_k(t)+\cdots+G_1(t)]. \qquad (9.8)$$

X. THE USE OF AN ANALOG COMPUTER FOR SOLVING THE DYNAMICAL EQUATIONS OF A LINE OF TRAFFIC

The special cases discussed above yield readily to mathematical analysis. However, if a complete study is desired based on complicated velocity functions of the leading vehicle, the analytical work may soon become quite formidable. In order to solve problems involving complicated velocities of the leading vehicle, it is suggested that the problem be attacked by the use of an electrical analog computer.

Electrical circuits in which the potential distribution satisfies the same differential equations as the dynamical equations of the line of traffic (2.5) are easy to construct or are commercially available.[18–19] Figure 8 illustrates a

[18] D. J. Mynall, "Electrical Analog Computing," Electronic Eng. (Brit.), Part 1, 178 (June, 1947); Part 2, 214 (July, 1947); Part 3, 259 (August, 1947); Part 4, 285 (September, 1947).

[19] G. D. McCann and H. E. Criner, Mach. Design (December, 1945, and February, 1946).

block diagram of a four-terminal network which is called a "Lagger" by exponents of electrical computation.

The Laplace transforms of the potential input and output of this four-terminal network are related by the equation

$$V_2(p)=\frac{1}{(Tp+1)}V_1(p). \tag{10.1}$$

This is identical with the first of the equations (3.5). It is thus apparent that by taking the "Lagger" network as a building block, a cascade connection of "Laggers" as shown in Fig. 9 can be connected.

The transform of the potential to the right of "Lagger" k is related to the transform of the potential applied to "Lagger" No. 1 by the equation

$$V_{k+1}=\frac{1}{(Tp+1)^k}V_1. \tag{10.2}$$

This is exactly the set of equations (3.6). It is thus seen that to study the motion of the line of traffic under different assumed types of motion of the leading vehicle, it is necessary to insert a potential to the left of the first unit of Fig. 9 and then to measure the potentials to the right of the several units by means of an oscilloscope or other suitable device.

XI. CONCLUSION

The analysis of a line of traffic presented in this discussion is based on postulates that are considerably removed from reality. However, in keeping with the spirit of "Operations Research" it was thought valuable to study the distribution of velocities and accelerations of the various vehicles which would result if the "following law" of the California Vehicle Code were obeyed by the operators of the vehicles at all times. It is hoped that by investigations of this type, eventually certain fundamental parameters involving vehicle and driver characteristics may be discovered that may in time be incorporated into the design of motor vehicles and roads to increase the ease of traffic flow and the safety of driving.

APPENDIX

TABLE I. Table of Laplace transforms. $g(p)=p\int_0^\infty e^{-pt}h(t)dt$.

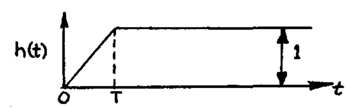
No.	$g(p)$	$h(t)$
1	1	Unit step function $1(t)=U(t)$
2	$g(Tp)$	$h(t/T)$
3	$e^{-pT}g(p)$	$U(t-T)h(t-T)$
4	$1/Tp$	t/T
5	$1/(Tp+1)$	$(1-e^{-t/T})$
6	$1/(Tp+1)^2$	$\left[1-e^{-t/T}\left(1+\frac{t}{T}\right)\right]$
7	$1/(Tp+1)^3$	$\left[1-e^{-t/T}\left(1+\frac{t}{T}+\frac{t^2}{2T^2}\right)\right]$
8	$1/(p+1)^k$	$G_k(t)\quad k>0$
9		$G_k(t)=1-e^{-t}\left[1+\frac{t}{1!}+\frac{t^2}{2!}+\cdots+\frac{t^{k-1}}{(k-1)!}\right]$ $k=1, 2, 3, \cdots$ $=\frac{1}{Q(k)}\int_0^t e^{-u}u^{(k-1)}du=\frac{Q_t(k)}{Q(k)}$
10		if $k>0$, $G_k(0)=0$, $G_k(\infty)=1$
11	$\frac{p}{(p+1)^k}$ $k=1, 2, 3, \cdots$	$G_k'(t)=\frac{e^{-t}t^{(k-1)}}{(k-1)!}=\Phi_k(t)$
12	$\frac{1}{p(p+1)^k}$	$\int_0^t G_k(u)du, \quad k=1, 2, 3, \cdots$
13	$\frac{Tp}{(Tp+1)}$	$e^{-t/T}$
14	$\frac{(1-e^{-Tp})}{Tp}$	

TABLE II. Short table of the function. $G_k(t)$.

			$G_k(t)$			
t	$G_1(t)$	$G_2(t)$	$G_3(t)$	$G_4(t)$	$G_5(t)$	$G_6(t)$
0	0	0	0	0	0	0
1	0.632	0.264	0.080	0.019	0.004	0.0006
2	0.865	0.594	0.323	0.143	0.053	0.016
3	0.950	0.800	0.577	0.353	0.185	0.084
4	0.981	0.908	0.762	0.566	0.371	0.215
5	0.993	0.959	0.875	0.735	0.559	0.384
6	0.999	0.983	0.938	0.849	0.715	0.554
7	0.999	0.993	0.970	0.918	0.827	0.699
8	1.000	0.997	0.986	0.957	0.900	0.809
9	1.000	0.999	0.994	0.979	0.945	0.884
10	1.000	1.000	0.997	0.989	0.970	0.933
∞	1	1	1	1	1	1