A SAFETY INDEX FOR TRAFFIC WITH LINEAR SPACING

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(Received March 5, 1959)

The authors have tried to improve the traffic dynamic theory of Professor Pipes and have found the fundamental equation of traffic dynamics for a case in which the velocity of the following vehicle is determined not only by the space between following and lead cars but also by the velocity of the lead car. The safety of the following car with respect to rear end collision when the lead car is in sinusoidal motion is quantified by introducing a safety index. At the same time, the authors describe the stability of the indicial response of the following car, and the stability of the propagation of a sinusoidal disturbance down a line of cars.

OUR STUDY starts by discussing the traffic dynamic theory of Professor PIPES,[1] and then goes on to propose a new fundamental equation which is deduced by introducing reaction time lag into his fundamental equation.[2] In our fundamental equation, we consider that the velocity of a following vehicle depends not only on the car space between the following and the lead car but also on the velocity of the lead car. The fundamental equation for a special case in which the velocity of the following vehicle depends only on the car space between the lead car coincides with the fundamental equation proposed independently of our study by R. E. CHANDLER, R. HERMAN, AND E. W. MONTROLL.[3]

A parameter α or m relating to the velocity of the lead car that appears in our consideration is important particularly when we investigate the traffic flow containing several kinds of cars whose braking abilities are widely different, as is observed on the highways or on the streets in Japan.

In this paper, the authors first show the fundamental equation of traffic dynamics, next discuss the stability of indicial response and that of the propagation of sinusoidal disturbance, and lastly describe the safety index. The safety index is a measure to represent the safety when the traffic flow is in sinusoidal motion.

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It will rarely be observed that the actual traffic flow behaves in a sinusoidal motion for a long period of time, but the behavior of very congested traffic disturbed by several sources such as the existence of bicycles, pedestrians, and parked cars may be considered approximately as a sinusoidal motion. These traffic disturbances that prevent driving at a uniform velocity can easily be observed in many city streets in Japan. In such traffic condition, safety is greatly reduced as compared to traffic flowing with uniform velocity. This is considered one reason for the high rate of rear end collisions in Japan despite the fact that the traffic volume is not large compared with that of the United States.

FUNDAMENTAL EQUATIONS OF TRAFFIC DYNAMICS

WE CONSIDER the manner in which vehicles of a single type follow each other in a queue on a highway without passing. When the following

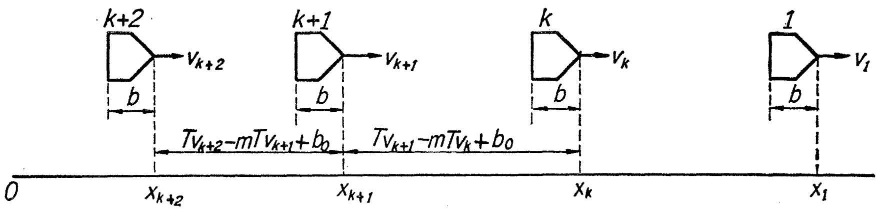

Fig. 1. Model of traffic flow.

vehicle drives so as to keep the car space behind the lead car to the minimum safe interval, the behavior of the lead car will control that of the following car completely.

We consider right bound traffic flow in Fig. 1. In general, the car space depends upon the driver's perception ability, so there is no assurance that the space he keeps is the safe car space. Even if the driver of the following car tries to follow the variations of behavior of the lead car, it is inevitable that there be a time lag of at least T, the reaction time. Now, let $x_k(t)$, $x_{k+1}(t)$ and $v_k(t)$, $v_{k+1}(t)$ be the coordinates and the velocities of the kth and $(k+1)$th vehicles respectively at a certain time t. Then, we have equation (1):

$$v_{k+1}(t)=f[x_k(t-T)-x_{k+1}(t-T),\, v_k(t-T)], \tag{1}$$

where f denotes a function defined by the experience of drivers. To simplify the analysis we deal with a case when the right hand side of equation (1) is a linear form. Namely,

$$x_k(t-T)-x_{k+1}(t-T)=\alpha\, v_k(t-T)+\beta\, v_{k+1}(t)+b_0, \tag{2}$$

where α, β, and b_0 are all constants. Equation (2) tells us that we are dealing with a case when the car space is expressed by a linear function of the velocities of the lead car and the following car. For convenience in the analysis we put

$$\alpha = -mT, \qquad \beta = nT. \qquad (m \geqq 0, n \geqq 0) \quad (3)$$

As α is always negative, we give it a minus sign. Differentiating equation (2), we get equation (4):

$$v_k(t-T) - v_{k+1}(t-T) = -mT\,\dot{v}_k(t-T) + nT\,\dot{v}_{k+1}(t). \qquad (4)$$

Equation (4) is the fundamental equation when the car space is expressed as a linear function of velocities. We have already discussed the case $m=0$,[2] and we notice here that the same form of equation for $m=0$ had been proposed by Chandler, Herman, and Montroll a little earlier and quite independently.

INDICIAL RESPONSE AND ITS STABILITY

IF WE make the Laplace transform of both sides of equation (4):

$$\begin{aligned} V_k(s)\,e^{-Ts} - V_{k+1}(s)\,e^{-Ts} \\ = -mTs\,V_k(s)\,e^{-Ts} + nTs\,V_{k+1}(s) + mT\,e^{-Ts}\,v_k(0) - nT\,v_{k+1}(0), \end{aligned}$$

or

$$V_{k+1}(s) = \frac{1+mTs}{nTs+e^{-Ts}} e^{-Ts}\,V_k(s) - \frac{mTe^{-Ts}}{nTs+e^{-Ts}}\,v_k(0) + \frac{nT}{nTs+e^{-Ts}} v_{k+1}(0), \quad (5)$$

where $\quad V_k(s) = \int_0^\infty v_k(t)\,e^{-ts}\,dt, \qquad V_{k+1}(s) = \int_0^\infty v_{k+1}(t)\,e^{-ts}\,dt,$

and $v_k(0)$ and $v_{k+1}(0)$ are the initial velocities at $t=0$. By equation (5) we can find the motion of the following vehicle if the motion of the lead car is given. Although we can consider several types of motion for the lead car, we take the simplest case. Namely all cars are standing when $t<0$, and the lead car starts to move at $t=0$ with its initial velocity v_0. In this paper, we hereafter call the behavior (output) of the following car corresponding to this behavior (input) of lead car as indicial response.

The initial conditions are $v_1(t) = v_0$ for $t \geqq 0$, or $V_1(s) = v_0/s$, $v_{k+1}(0) = 0$ $(k=1, 2, 3, \cdots)$. Therefore, using equation (5) repeatedly we find:

$$\begin{aligned} V_2 &= [e^{-Ts}/(nTs+e^{-Ts})](v_0/s), \\ V_3 &= [e^{-Ts}/(nTs+e^{-Ts})]^2\,(1+mTs)(v_0/s), \\ &\cdots \qquad \cdots \qquad \cdots \\ V_{k+1} &= [e^{-Ts}/(nTs+e^{-Ts})]^k\,(1+mTs)^{k-1}\,(v_0/s). \end{aligned} \qquad (6)$$

Expanding equation (6) in the power series to compute v_2, v_3, we get

$$\frac{V_2(s)}{v_0}=\frac{e^{-Ts}}{nTs}-\frac{e^{-2Ts}}{n^2T^2s^2}+\frac{e^{-3Ts}}{n^3T^3s^3}-\frac{e^{-4Ts}}{n^4T^4s^4}+\frac{e^{-5Ts}}{n^5T^5s^5}-\cdots,$$

$$\frac{V_3(s)}{v_0}=\frac{e^{-2Ts}}{n^2T^2s^3}-\frac{2\,e^{-3Ts}}{n^3T^3s^4}+\frac{3\,e^{-4Ts}}{n^4T^4s^5}-\frac{4\,e^{-5Ts}}{n^5T^5s^6}+\cdots \tag{7}$$

$$+mT\left[\frac{e^{-2Ts}}{(nTs)^2}-\frac{2\,e^{-3Ts}}{(nTs)^3}+\frac{3\,e^{-4Ts}}{(nTs)^4}-\frac{4\,e^{-5Ts}}{(nTs)^5}+\cdots\right].$$

Therefore the indicial responses required are given by

$$\frac{v_2(t)}{v_0}=\frac{1}{n}\left(\frac{t}{T}-1\right)-\frac{1}{2\,n^2}\left(\frac{t}{T}-2\right)^2+\frac{1}{6\,n^3}\left(\frac{t}{T}-3\right)^3-\frac{1}{24\,n^4}\left(\frac{t}{T}-4\right)^4$$

$$+\frac{1}{120\,n^5}\left(\frac{t}{T}-5\right)^5-\frac{1}{720\,n^6}\left(\frac{t}{T}-6\right)^6+\frac{1}{5040\,n^7}\left(\frac{t}{T}-7\right)^7-\cdots$$

$$\frac{v_3(t)}{v_0}=\frac{1}{2\,n^2}\left(\frac{t}{T}-2\right)^2-\frac{1}{3\,n^3}\left(\frac{t}{T}-3\right)^3+\frac{1}{8\,n^4}\left(\frac{t}{T}-4\right)^4-\frac{1}{30\,n^5}\left(\frac{t}{T}-5\right)^5 \tag{8}$$

$$+\frac{1}{144\,n^6}\left(\frac{t}{T}-6\right)^6-\frac{1}{840\,n^7}\left(\frac{t}{T}-7\right)^7+\cdots$$

$$+m\left[\frac{1}{n^2}\left(\frac{t}{T}-2\right)-\frac{1}{n^3}\left(\frac{t}{T}-3\right)^2+\frac{1}{2\,n^4}\left(\frac{t}{T}-4\right)^3-\frac{1}{6\,n^5}\left(\frac{t}{T}-5\right)^4\right.$$

$$\left.+\frac{1}{24\,n^6}\left(\frac{t}{T}-6\right)^5-\frac{1}{120\,n^7}\left(\frac{t}{T}-7\right)^6+\cdots\right].$$

As seen in equation (8), $v_2(t)$ is independent of m. Figure 2 shows the indicial responses computed from equation (8) for several values of m and n. As shown in Fig. 2 the amplitude of response becomes larger as m becomes larger or n smaller.

Next, we consider the response of a following vehicle when the lead car of a line of vehicles that are moving with uniform velocity v_0 stops suddenly at $t=0$. The initial conditions are $v_1(t)=0$ for $t\geqq 0$, $v_{k+1}(0)=v_0$ $(k=1, 2, 3, \cdots)$; therefore

$$\begin{aligned} V_2&=F(s)\,v_0,\\ V_3&=E(s)\,V_2+D(s)\,v_0=E(s)\,F(s)\,v_0+D(s)\,v_0,\\ &\cdots\qquad\cdots\qquad\cdots\\ V_{k+1}&=E(s)\,V_k+D(s)\,v_0\\ &=E^{k-1}(s)\,F(s)\,v_0+E^{k-2}(s)\,D(s)\,v_0+\cdots+D(s)\,v_0, \end{aligned} \tag{9}$$

where

$$\begin{aligned} D(s)&=(nT-mT\,e^{-Ts})/(nTs+e^{-Ts}),\\ E(s)&=(1+mTs)\,e^{-Ts}/(nTs+e^{-Ts}),\\ F(s)&=nT/(nTs+e^{-Ts}). \end{aligned}$$

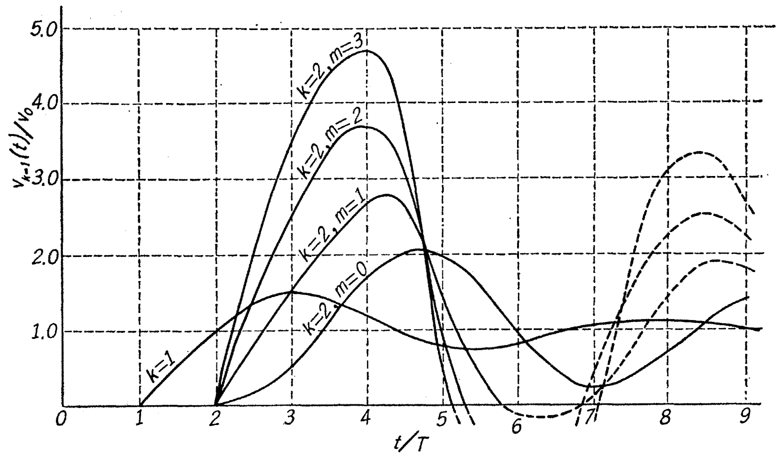

Fig. 2 (a). Indicial response of the following vehicles for $n=1$.

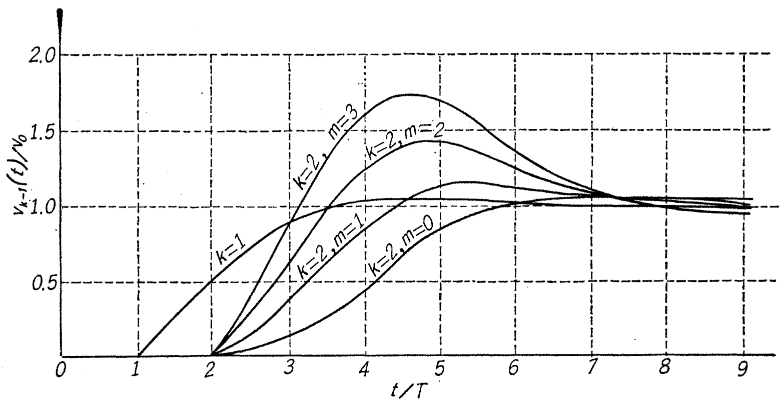

Fig. 2 (b). Indicial response of the following vehicles for $n=2$.

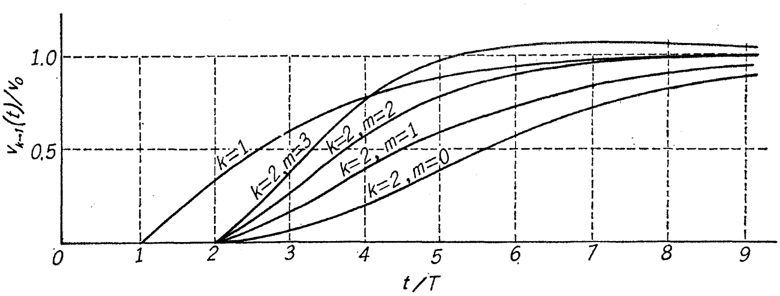

Fig. 2 (c). Indicial response of the following vehicles for $n=3$.

We can see from equation (9) that V_2 is independent of m. If we designate the two velocities V in equation (6) and equation (9) as V_{start} and V_{stop},

$$
\begin{aligned}
V_{k+1,\text{stop}}/v_0 &= E^{k-1}(s)\,F(s)\,v_0 \\
&\qquad + D(s)\,[E^{k-2}(s)+E^{k-3}(s)+\cdots+E(s)+1] \\
&= E^{k-1}(s)\,[D(s)+mT\,G(s)] \\
&\qquad + D(s)\,[E^{k-2}(s)+E^{k-3}(s)+\cdots+E(s)+1] \\
&= mT\,G(s)\,E^{k-1}(s)+D(s)\,[1-E^{k}(s)]/[1-E(s)] \\
&= mT\,G(s)\,E^{k-1}(s)+[1-E^{k}(s)]/s \qquad (10) \\
&= G(s)\,E^{k-1}(s)/s+mT\,G(s)\,E^{k-1}(s) \\
&\qquad + [1-E^{k}(s)]/s-V_{k+1,\text{start}}/v_0 \\
&= [(1+mTs)/s]\,G(s)\,E^{k-1}(s) \\
&\qquad + 1/s-E^{k}(s)/s-V_{k+1,\text{start}}/v_0 \\
&= 1/s-V_{k+1,\text{start}}/v_0.
\end{aligned}
$$

The inverse transform of equation (10) gives

$$v_{k+1,\text{start}}(t)+v_{k+1,\text{stop}}(t)=v_0. \qquad (k=1,2,3,\cdots) \quad (11)$$

If we can find either one of v_{start} or v_{stop}, we can calculate the other by equation (11).

Next we consider the stability of indicial response. If the indicial response approaches its final value v_0 as time elapses, the response is stable,

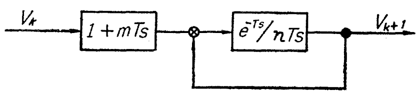

Fig. 3. Block diagram of a queue of traffic.

but if the indicial response shows a permanent oscillation, the response is unstable.

As the transfer function of equation (5) is given by

$$E(s)=e^{-Ts}\,(1+mTs)/(nTs+e^{-Ts}), \qquad (12)$$

its block diagram is shown as a servo system as in Fig. 3. If the characteristic equation of equation (12),

$$nTs+e^{-Ts}=0, \qquad (13)$$

is assumed to have the following type of roots,

$$s_i=-\sigma_i+j\omega_i, \qquad (i=1,2,3,\cdots) \quad (14)$$

then these characteristic roots can be calculated from

$$\sigma_i = \frac{\omega_i}{\tan\omega_i T}, \qquad \omega_i = \pm\sqrt{\frac{e^{2\sigma_i T}}{(nT)^2} - \sigma_i^2}. \tag{15}$$

The indicial response of the second vehicle (just behind the lead car) is

$$\begin{aligned} v_2(t) &= \frac{v_0}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{(e^{-Ts})(e^{st})}{s\,(nTs+e^{-Ts})}\,ds \\ &= v_0 \sum \text{residues of } [P(s)\,e^{st}/Q(s)], \end{aligned} \tag{16}$$

where $P(s)=e^{-Ts}$, $Q(s)=s(nTs+e^{-Ts})$. Since $e^{st}P(s)/Q(s)$ is regular except for $s=0$ and $s=\mathcal{S}_i$, its residue is unity at the simple pole $s=0$, and the residues at the simple pole $s=\mathcal{S}_i$ are described by

$$e^{\mathcal{S}_i t}\,P(\mathcal{S}_i)/Q'(\mathcal{S}_i) = [-1/(1+T\mathcal{S}_i)]\,e^{\mathcal{S}_i t}. \qquad (i=1,2,3,\cdots)$$

Hence we have from equation (16)

$$v_2(t) = v_0\,[1+\sum_{i=1}^{i=\infty} -1/(1+T\mathcal{S}_i)\,e^{\mathcal{S}_i t}]. \tag{17}$$

If we put

$$R_i = -1/(1+T\mathcal{S}_i), \tag{18}$$

corresponding to a pair of conjugate imaginary roots

$$\mathcal{S}_i = -\sigma_i \pm j\omega_i, \tag{19}$$

we have

$$v_2(t) = v_0\,[1+2\sum |R_i|\,e^{-\sigma_i t}\cos(\omega_i t+\arg R_i]. \tag{20}$$

If we denote by $\mathcal{S}_1$ the nearest root from the imaginary axis of all the characteristic roots, the response is represented approximately by a component response of $\mathcal{S}_1$, that is,

$$v_2(t) = v_0\,[1+2\,|R_1|\,e^{-\sigma_1 t}\cos(\omega_1 t+\arg R_1)]. \tag{21}$$

Equation (21) coincides well for $t/T>1$ with the solution obtained by the above mentioned power series method.

As shown in equation (20), the real parts of all the characteristic roots should be negative when the indicial response of the second vehicle is stable. Since $\sigma_1=0$ at the stability limit, we can derive directly from equation (15) the condition defining the limit of stability,

$$n = 2/\pi. \tag{22}$$

On the other hand, the critical condition in which oscillation disappears may be similarly derived from $\omega_1=0$, that is

$$n = e. \tag{23}$$

Next we consider the general case. The indicial response for $(k+1)$th vehicle is given from equation (6) as

$$\begin{aligned} v_{k+1}(t) &= \frac{v_0}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^{-kTs}\,(1+mTs)^{k-1}}{s\,(nTs+e^{-Ts})^k}\,e^{st}\,ds \\ &= v_0 \sum \text{residues of } [K(s)\,e^{st}], \end{aligned} \tag{24}$$

where $K(s)=e^{-kTs}\,(1+mTs)^{k-1}/s(nTs+e^{-Ts})^k$.

As the characteristic equation for $K(s)$ is

$$s\,(nTs+e^{-Ts})^k=0, \tag{25}$$

we have in general

$$\begin{aligned} \frac{v_{k+1}(t)}{v_0} &= 1+\sum_{i=1}^{\infty}\left\{\frac{t^{k-1}\,e^{\mathcal{S}_i t}}{(k-1)!}\,[(s-\mathcal{S}_i)\,K(s)]_{s=\mathcal{S}_i}\right. \\ &+\frac{t^{k-2}\,e^{\mathcal{S}_i t}}{(k-2)!}\left[\frac{d}{ds}(s-\mathcal{S}_i)\,K(s)\right]_{s=\mathcal{S}_i}+\frac{t^{k-3}\,e^{\mathcal{S}_i t}}{2!\,(k-3)!}\left[\frac{d^2}{ds^2}(s-\mathcal{S}_i)K(s)\right]_{s=\mathcal{S}_i} \\ &\left.+\cdots+\frac{e^{\mathcal{S}_i t}}{(k-1)!}\left[\frac{d^{k-1}}{ds^{k-1}}(s-\mathcal{S}_i)\,K(s)\right]_{s=\mathcal{S}_i}\right\}. \end{aligned} \tag{26}$$

As is known from equation (26), the indicial response has some component responses of the form $t^{k-1}e^{-\sigma_i t}$ when the characteristic equation has the root of kth order. As long as $\sigma_i>0$ these component responses will be damped as time elapses.

Since the indicial responses may be represented approximately in terms of $\mathcal{S}_1$ as mentioned above, the stability condition in general is $n>2/\pi$, and the critical condition in which oscillation disappears is $n=e$. Hence we have the following conclusion: *If a car space is given by the linear form of both velocities of the lead car and the following car, the indicial response of each vehicle in a line of traffic is unstable for $n\leqq 2/\pi$, stable with oscillation for $2/\pi<n<e$, and stable without oscillation for $n\geqq e$.*

Although we have already had the above conclusion for the case of $m=0$, we have found here that m does not affect the stability of the indicial response because m does not appear in the characteristic equation. As can be supposed from equation (5) and equation (24), the above mentioned conclusion will hold not only in the case of indicial response but also for all the behavior of the lead car provided that $V_1(s)$ has no characteristic root except at $s=0$, namely, $sV_1(s)$ is regular. This is because the characteristic root which controls the stability of transient response is the same. It is important that the stability condition of transient response is independent of m and depends on n only.

FREQUENCY RESPONSE AND STABILITY OF PROPAGATION OF A SINUSOIDAL DISTURBANCE

HERE WE investigate the manner in which the disturbance propagates down a line of vehicles when the lead car suffers a sinusoidal disturbance. We call it unstable if the disturbance of the lead car propagates down magnified, and call it stable if it propagates down damped.

As is known from equation (5), the transfer function of $(k+1)$th vehicle is given by

$$E_{k+1}(s)=E^{k}(s)=[e^{-Ts}\ (1+mTs)/(nTs+e^{-Ts})]^{k}. \qquad (27)$$

Therefore, if the disturbance suffered by the lead car is expressed by

$$a=A\ \sin\omega t, \qquad (28)$$

the frequency response of $(k+1)$th vehicle after elapse of sufficient time is

$$z=Z\ \sin(\omega t+\varphi), \qquad (29)$$

where

$$\begin{aligned} Z&=A\ |E_{k+1}(j\omega)|=A\ \{|E(j\omega)|\}^{k}, \\ \varphi&=\tan^{-1}\mathrm{Im}\ [E_{k+1}(j\omega)]/\mathrm{Re}\ [E_{k+1}(j\omega)]. \end{aligned} \qquad (30)$$

The initial sinusoidal oscillation of the lead car is transmitted to the following vehicles magnified if $|E(j\omega)|>1$ and is transmitted damped if $|E(j\omega)|<1$. Hence the stability condition of the propagation is $|E(j\omega)|<1$, or more explicitly

$$|e^{-j\omega T}\ (1+jm\omega T)/(jn\omega T+e^{-j\omega T})|<1. \qquad (31)$$

Since we have

$$\begin{aligned} |E(j\omega)|&=\sqrt{\frac{1+m^2\omega^2T^2}{1+n^2\omega^2T^2-2\ n\omega T\ \sin\omega T}}, \\ \varphi&=\tan^{-1}\frac{m\omega T-n\omega T\ \cos\omega T-mn\omega^2T^2\ \sin\omega T}{1-n\omega T\ \sin\omega T+mn\omega^2T^2\ \cos\omega T}, \end{aligned} \qquad (32)$$

the stability condition is given by

$$(n^2-m^2)/n>2\ \sin\omega T/\omega T. \qquad (33)$$

As has already been investigated, the stability condition in case of $m=0$ is given by $n>2\ \sin\omega T/\omega T$, so the condition for stability for all ωT is $n>2$. This is a special case of equation (33). As $\sin\omega T/(\omega T)$ is less than unity for all ωT, the sufficient condition for stability for a disturbance of any frequency is given by

$$n>1+\sqrt{1+m^2}. \qquad (34)$$

The interesting case is the one for $n=m$. In this case, as is known from equation (33) the stability condition exists quite independent of m and n, and is stable if $\sin\omega T<0$, namely

$$(2i-1)\,\pi<\omega T<2\,i\pi. \qquad (i=1, 2, 3, \cdots) \quad (35)$$

It is always unstable for ωT which does not satisfy equation (35) provided $\omega T\geqq 0$.

Figure 4 shows the amplitude of the frequency responses computed from equation (32) for several values of m and n. The case for $n=m$ is illustrated in Fig. 5. As is known from Figs. 4 and 5, the amplitudes of the frequency response become larger if m becomes larger and n becomes smaller. In general, we find that disturbances of high frequency are not propagated down a line of vehicles.

Clearly, it follows from equation (35) that each response in Fig. 5 coincides at a point C when $\omega T=\pi$, and the left side of the point C is the unstable domain.

SAFETY INDEX

Let us consider that a line of vehicles is moving in one direction with a uniform speed v_0 and the lead car has suffered a sinusoidal disturbance at $t=0$. The behavior of the lead car is expressed by

$$v_1(t)=\begin{cases} v_0-A\,\sin\omega t & \text{for} \quad t\geqq 0, \\ v_0 & \text{for} \quad t<0, \end{cases} \qquad (36)$$

where $v_0\geqq A$.

The behavior of the lead car is propagated to the following vehicles with the lag of reaction time T. After a sufficient time, the $(k+1)$th vehicle reaches a steady state motion expressed by

$$v_{k+1}(t)=v_0-A\,|E_{k+1}(j\omega)\,\sin|\,\omega t+\arg E_{k+1}(j\omega)]. \qquad (37)$$

We will investigate the safety of cars under this steady state condition. When all vehicles are moving with a uniform speed v_0 the car space is expressed by $(n-m)\,Tv_0+b_0$ as is known from Fig. 1, so that the car space $\gamma(t)$ between the lead car and the second car at a time t after the lead car has begun to behave as shown in equation (36) is given by

$$\gamma(t)=(n-m)\,Tv_0+b_0+\int_0^t v_1(t)\,dt-\int_0^t v_2(t)\,dt, \qquad (38)$$

where $\qquad \lim_{t\to 0}\gamma(t)=(n-m)\,Tv_0+b_0.$

If a line of vehicles is in such a steady state condition and the lead car is suddenly stopped by any obstacle, the driver of the second car will apply the brakes promptly. But if the car space is not greater than

$$\int_t^{t+T} v_2(t)\,dt+\mu_2\,v_2^{\,2}(t+T)-\mu_1\,v_1^{\,2}(t)+b, \qquad (39)$$

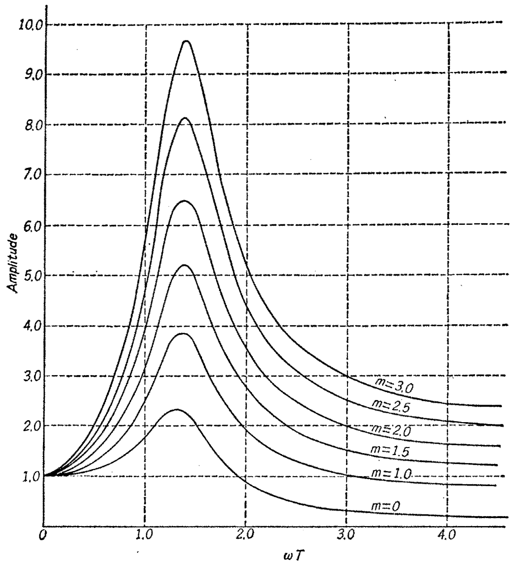

Fig. 4 (a). Amplitude of frequency response for $n=1$.

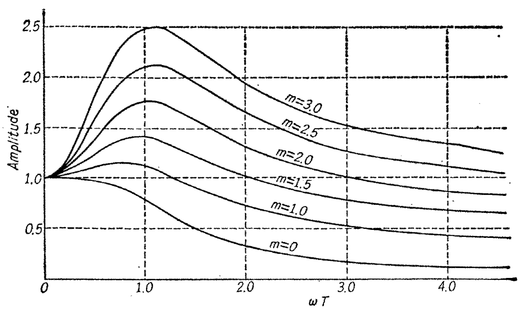

Fig. 4 (b). Amplitude of frequency response for $n=2$.

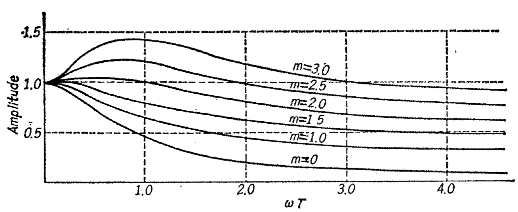

Fig. 4 (c). Amplitude of frequency response for $n=3$.

the second car cannot escape a rear end collision with the lead car. We define μ_1 and μ_2 to be the deceleration of the lead car and the second car respectively, T the reaction time, and b the car length.

We are considering the steady state condition, v_1 and v_2 in equation (39) as given in equations (36) and (37). But we cannot use equation (37) for v_2 because the third term of equation (38) has an integration whose lower limit is zero. Therefore we suffer from the effect of the transient response of v_2, and must first find

$$\Gamma(t)=\int_0^t v_1\,dt-\int_0^t v_2\,dt, \tag{40}$$

where $\lim_{t\to 0}\Gamma(t)=0.$

If we express the Laplace transform of $\Gamma(t)$ as $\Gamma(s)$,

$$\Gamma(s)=[V_1(s)-V_2(s)]/s, \tag{41}$$

from equation (5) the following relation exists:

$$V_2(s)=\frac{e^{-Ts}\,(1+mTs)}{nTs+e^{-Ts}}\,V_1(s)-\frac{nT\,e^{-Ts}}{nTs+e^{-Ts}}\,v_1(0)+\frac{nT}{nTs+e^{-Ts}}\,v_2(0).$$

Therefore

$$\Gamma(s)=\frac{nT-mT\,e^{-Ts}}{nTs+e^{-Ts}}\,V_1(s)+\frac{mTe^{-Ts}}{s\,(nTs+e^{-Ts})}\,v_1(0)-\frac{nT}{s\,(nTs+e^{-Ts})}\,v_2(0). \tag{42}$$

Considering the following initial conditions:

$$v_1(0)=v_2(0)=v_0,\qquad V_1(s)=(v_0/s)-(A\omega/s^2+\omega^2), \tag{43}$$

and substituting equation (43) into equation (42), we can get

$$\Gamma(s)=-\frac{A\omega\,(nT-mTe^{-Ts})}{(s^2+\omega^2)(nTs+e^{-Ts})}\equiv-\mathcal{P}(s)/\mathcal{Q}(s). \tag{44}$$

The characteristic roots of equation (44) are $s=\mathcal{S}_i$, which are given by equation (14), equation (15), and $s=\pm j\omega$, the roots of $s^2+\omega^2=0$.

The Laplace transform of equation (44) is given by

$$\Gamma(t)=-\frac{A\omega}{2\pi j}\int_{-j\infty}^{j\infty}\frac{nT-mT\,e^{-Ts}}{(s^2+\omega^2)(nTs+e^{-Ts})}\,e^{st}\,ds. \tag{45}$$

The residue of $e^{st}\,\mathcal{P}(s)/\mathcal{Q}(s)$ at $s=\mathcal{S}_i$ is

$$e^{\mathcal{S}_i t}\,\mathcal{P}(\mathcal{S}_i)/\mathcal{Q}'(\mathcal{S}_i)=A\omega\,\frac{1+mT\mathcal{S}_i}{(\mathcal{S}_i^2+\omega^2)(1+T\mathcal{S}_i)}\,e^{\mathcal{S}_i t},$$

and therefore the term concerning the characteristic root s_i that will appear in computing equation (45) diminishes as time elapses provided $\sigma_i > 0$. On the other hand, the residue of $e^{st}\,\mathcal{P}(s)/\mathcal{Q}(s)$ at a simple pole $s=j\omega$ is given by

$$R\,e^{j\omega t}=AT\frac{n\,(-n\omega T+\sin\omega T+m\omega T\cos\omega T)+j\,(m-mn\omega T\sin\omega T-n\cos\omega T)}{2(1+n^2\omega^2T^2-2\,n\omega T\sin\omega T)}\,e^{j\omega t}, \tag{46}$$

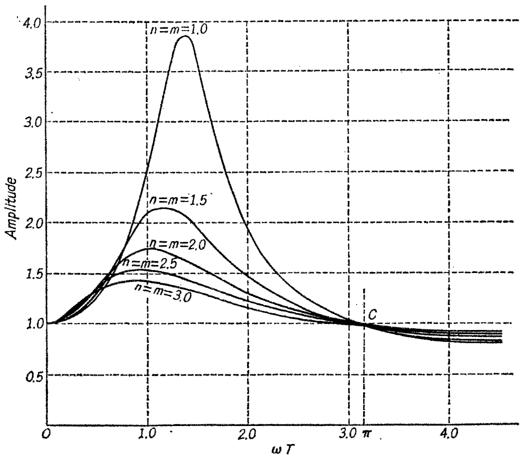

Fig. 5. Amplitude of frequency response for $n=m$.

and the residue at $s=-j\omega$ is

$$R'\,e^{-j\omega t}=AT\frac{n\,(-n\omega T+\sin\omega T+m\omega T\cos\omega T)-j\,(m-mn\omega T\sin\omega T-n\cos\omega T)}{2(1+n^2\omega^2T^2-2n\omega T\sin\omega T)}\,e^{-j\omega t}. \tag{47}$$

Since the following relations exist:

$$|R|=|R'|,\qquad \arg R=-\arg R', \tag{48}$$

rearranging equations (46) and (47) we can get:

$$\Gamma(t)=-2\,|R|\cos(\omega t+\arg R). \tag{49}$$

Putting $$U=2\,R/AT, \tag{50}$$

$$\Gamma(t)=-AT\,|U|\cos(\omega t+\arg U), \tag{51}$$

where
$$|U|=\sqrt{\frac{n^2+m^2-2\,mn\cos\omega T}{1+n^2\omega^2T^2-2\,n\omega T\sin\omega T}},$$
$$\arg U=\tan^{-1}\frac{m-mn\omega T\sin\omega T-n\cos\omega T}{n(-m\omega T+\sin\omega T+m\omega T\cos\omega T)}. \tag{52}$$

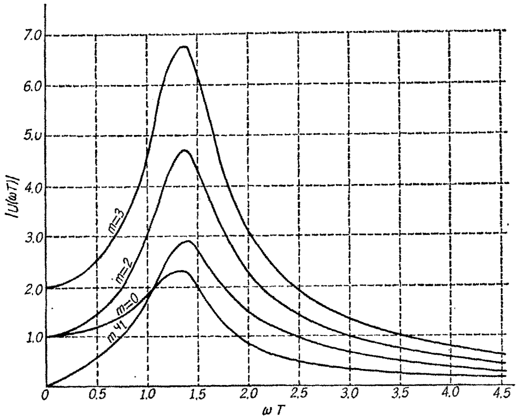

Fig. 6 (a). Amplitude of fluctuation of a car spacing from constant spacing for $n=1$.

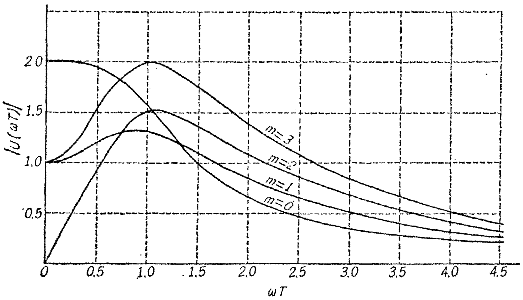

Fig. 6 (b). Amplitude of fluctuation of a car spacing from constant spacing for $n=2$.

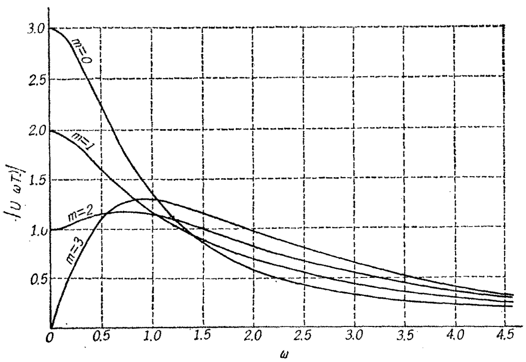

Fig. 6 (c). Amplitude of fluctuation of a car spacing from constant spacing for $n=3$.

Figure 6 illustrates the values of $|U|$ for several m and n computed by equation (52). It is shown that the maximum amplitude of fluctuation increases as n becomes smaller or as the difference between n and m becomes larger. From equation (52) can be found the following relation

$$\lim_{\omega T\to 0} |U(\omega T)| = n-m. \tag{53}$$

Hence we obtain, by substituting equation (51) into equation (38),

$$\gamma(t) = (n-m)\ Tv_0 + b_0 - AT\ |U|\ \cos(\omega t + \arg U). \tag{54}$$

On the other hand, we have the following relation from equation (37):

$$\int_t^{t+T} v_2\, dt = v_0\, T - ATW\ \sin(\omega t + \varphi'), \tag{55}$$

where

$$\begin{aligned} W &= (2/\omega T)\ |E|\ \sin(\tfrac{1}{2}\,\omega T), \\ \varphi' &= \tfrac{1}{2}\,\omega T + \varphi. \end{aligned} \tag{56}$$

Therefore, if the lead car stopped suddenly during steady state sinusoidal fluctuations, the necessary condition for the following vehicle not to collide with the rear of the lead car is given from equations (54) and (39) as follows:

$$\begin{aligned} &(n-m-1)\ Tv_0 + b_0 - b - AT\ |U|\ \cos(\omega t + \arg U) \\ &\qquad + ATW\ \sin(\omega t + \varphi') > \mu_2 v_2^2(t+T) - \mu_1\, v_1^2(t). \end{aligned} \tag{57}$$

If we put $\quad l_0 - l = (b_0 - b)/T, \qquad \mu_1' = \mu_1/T, \qquad \mu_2' = \mu_2/T, \qquad (58)$

equation (57) becomes

$$\begin{aligned} &(n-m-1)\ v_0 + l_0 - l - A\ |U|\ \cos(\omega t + \arg U) \\ &\qquad + AW\ \sin(\omega T + \varphi') + \mu_1'\ v_1^2(t) - \mu_2'\ v_2^2(t+T) > 0. \end{aligned} \tag{59}$$

Developing equation (59) for convenience of computation, we have

$$\begin{aligned} &(n-m-1)\ v_0 + l_0 - l + A\ (\Delta_1 \cos\omega t + \Delta_2 \sin\omega t) + \mu_1'\ [v_0 - A\ \sin\omega t]^2 \\ &\qquad - \mu_2'[v_0 - A\ (\theta_1 \cos\omega t + \theta_2 \sin\omega t)]^2 > 0, \end{aligned} \tag{60}$$

where

$$\begin{aligned} \Delta_1 &= W\ \sin(\tfrac{1}{2}\,\omega T + \varphi) - |U|\ \cos(\arg U), \\ \Delta_2 &= W\ \cos(\tfrac{1}{2}\,\omega T + \varphi) + |U|\ \sin(\arg U), \\ \theta_1 &= |E|\ \sin(\omega T + \varphi), \\ \theta_2 &= |E|\ \cos(\omega T + \varphi). \end{aligned} \tag{61}$$

Here Δ_1, Δ_2, θ_1, and θ_2 are all the function of ωT and have the following properties:

$$\begin{aligned} &\lim_{\omega T\to 0} (\Delta_1,\ \theta_1) = 0, && \lim_{\omega T\to 0} \theta_2 = 1, \\ &\lim_{\omega T\to 0} \Delta_2 = n-m+1, && \lim_{\omega T\to \infty} (\Delta_1,\ \Delta_2,\ \theta_1,\ \theta_2) = 0. \end{aligned} \tag{62}$$

If the equation (59) or equation (60) always exists, the traffic flow is safe for sinusoidal disturbance. We denote the probability for the existence of equation (59) or equation (60) at an arbitrary time t as the safety index for the traffic flow, defined as follows:

$$\text{safety index} = \Pr[t; \text{equation (60)}]. \tag{63}$$

Figure 7 shows a numerical example for a case of $v_0 = 40$ km/h, $A = 20$ km/h, $l_0 - l = -2.2124$ m/sec provided that $\mu_1' = \mu_2' = \mu'$.

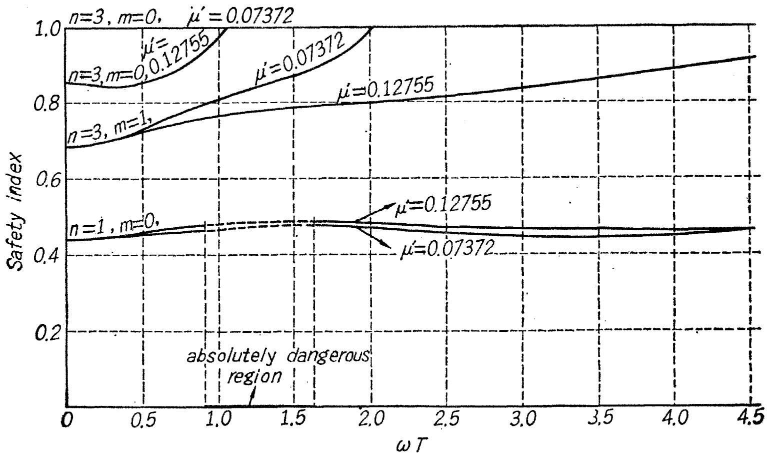

Fig. 7. Safety index varying with ωT ($v_0 = 40$ km/h, $A = 20$ km/h, $T = 1.13$ sec, $l_0 - l = -2.2124$ m/sec).

The value of the safety index for $\omega T \to 0$ is given from equation (62):

$$\Pr[\sin\omega t > -\{(n-m-1)\,v_0 + l_0 - l\}/\{(n-m+1)\,A\}, \tag{64}$$

whereas the value of the safety index for $\omega T \to \infty$ is given from

$$\Pr\left[\sin\omega t < \frac{v_0}{A}\left(1 - \sqrt{1 - \frac{n-m-1}{\mu' v_0} - \frac{l_0 - l}{\mu' v_0^2}}\right)\right]. \tag{65}$$

Here the following relation is assumed: $\mu' v_0^2 > (n-m-1)\,v_0 + l_0 - l$. As is known from equations (64) and (65), the values of the safety indices when $\omega T \to 0$ and $\omega T \to \infty$ depend remarkably on the value of $n-m$, and the safety index when $\omega T \to 0$ is independent of μ'. The dotted area in Fig. 7 is a domain where the safety index has no meaning, because a rear end collision certainly will occur. The condition of existence for these absolutely dangerous frequency ωT is given by equations (54) and (58):

$$(n-m)\,v_0 + l_0 - l - A\,|U(\omega T)| \leqq 0. \tag{66}$$

CONCLUSION

SOME properties of the safety index as suggested from Fig. 7 are:

1. The safety is determined almost completely by the difference between n and m, and the smaller the difference, the less the effect of braking ability and ω on the safety index. Conversely, as the difference becomes larger the safety index increases rapidly and it becomes necessary to depend on the braking ability and ω.

2. Except for extraordinarily large values of μ, the safety index is generally likely to increase with an increase of ωT. But the values of ω for the disturbances suffered in actual traffic flow are considered to vary between 0.1 and 1.5, so that it will be sufficient in actual traffic to deal with the safety index for relatively small values of ω.

3. Since the safety index is a proposal to quantify the safety of rear end collision when the lead car and the following car behave with sinusoidal motion, it is required for safety that ωT not satisfy equation (66).

IT IS A great pleasure to acknowledge our thanks to various members of the Operations Research Group at Case Institute of Technology for discussions on this subject. We are indebted in particular to DR. ROBERT HERMAN at the Research Laboratory of General Motors Corporation and PROFESSOR RENFREY POTTS of the University of Toronto for discussions and suggestions.

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